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On a conjecture of Shanks ☆

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ABSTRACT

The conjecture in question concerns the function ϕ_n related to the distribution of the zeroes of the Riemann zeta-function, γ_n, over the Gram points g_n. It is the purpose of this article to show that for any $\alpha > 0$ the sum

$$\frac{\sum_{n=1}^{K} \phi_n}{K^\alpha} \to 0,$$

and this was conjectured, on numerical evidence, by Shanks (1961) [7] to be true for $\alpha = \frac{1}{2}$.

1. Introduction

One of the first large scale numerical calculations relating to the Riemann zeta-function was conducted by Haselgrove [3] in 1960. Included in these tables are calculations of $\Re\zeta(\frac{1}{2}+it)$, $\Im\zeta(\frac{1}{2}+it)$ and of the functions $Z(t)$ and $\theta(t)$ defined by

$$Z(t) = e^{i\theta(t)}\zeta\left(\frac{1}{2}+it\right).$$

It can be shown (see, e.g. [9, §4] or [1, §§6–8]) that $Z(t)$ is real for real-valued t and that $\theta(t)$ is ultimately increasing. Therefore when $n \geqslant -1$, one defines the **Gram points** g_n, to be such that $\theta(g_n) = n\pi$. Of interest in the location of the zeroes of the zeta-function is **Gram's Law**, which states that each Gram interval[1] $(g_{n-1}, g_n]$ contains exactly one zero of $\zeta(\frac{1}{2}+it)$. Titchmarsh showed [8]

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[1] The convention in numbering the Gram points is that the first zero of $\zeta(\frac{1}{2}+it)$ at $t \approx 14.134\ldots$ lies in the interval $(g_{-1}, g_0]$.

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that Gram's Law fails infinitely often and results of the author [10] show, *inter alia* that a positive proportion of Gram intervals do not contain a zero.

Table III in [3] includes values of g_n, γ_n and ϕ_n, where γ_n denotes the ordinate of the nth zero on the critical line (counted with multiplicity) — in particular, the Riemann Hypothesis is not assumed. The numbers ϕ_n are defined by

$$\phi_n = n - \frac{3}{2} - \pi^{-1}\theta(\gamma_n), \tag{1}$$

and Shanks [7] states that Gram's Law fails whenever $|\phi_n| > \frac{1}{2}$. Care needs to be taken, since Shanks writes

$$\phi_n = \pi^{-1}\arg\zeta'\left(\frac{1}{2} + i\gamma_n\right),$$

as a definition for ϕ_n, and clearly the argument needs to be specified up to a multiple of 2π. But Shanks's statement easily follows from (1), since, if Gram's Law is true for all $m \leqslant n$, then $n - 2 < \pi^{-1}\theta(\gamma_n) \leqslant n - 1$.

Shanks gave some numerical data concerning the average of the sum $\sum_{n=1}^{K}\phi_n$. He conjectured that $(1/K)\sum_{n=1}^{K}\phi_n \to 0$, and, in a note added in proof correction, that the stronger estimate $(1/\sqrt{K})\sum_{n=1}^{K}\phi_n \to 0$ may hold. His paper contains no *prima facie* reason, other than the numerical evidence, to suggest why, if the latter conjecture were true, a similar result of the type $(1/K^{\alpha})\sum_{n=1}^{K}\phi_n \to 0$ might not also be true, for some $\alpha < \frac{1}{2}$. It is the object of this paper to answer both of Shanks's conjectures in the affirmative by proving a

Theorem. *If* $\alpha > 0$ *then*

$$\frac{\sum_{n=1}^{K}\phi_n}{K^{\alpha}} \to 0. \tag{2}$$

This conjecture of Shanks is not well known: there is a reference contained in [4, pp. 86–90]; the conjecture that $(1/K)\sum_{n=1}^{K}\phi_n \to 0$ is proved in [2]. Nevertheless the function ϕ_n is closely related to the function $\Delta(n)$ which has been studied by Titchmarsh [8] and by Selberg [6]. Define $\Delta(n) = n - m$, where γ_n lies in the mth Gram interval $(g_{m-2}, g_{m-1}]$, whence it follows from (1) that $|\phi_n - \Delta(n)| \leqslant \frac{1}{2}$ whenever Gram's Law holds up to n. Thus $\Delta(n)$ is a measure of how far the zeroes are 'out of sync' with the Gram points; indeed if Gram's Law were to hold universally, then $\Delta_n \equiv 0$. The function $\Delta(n)$ is very similar to the argument function $S(t)$ (properties of which can be found in [9, §9]) and so the sum considered by Shanks in (2) can be compared with $\int_0^T S(t)\,dt$; the proof is achieved using estimates of this integral.

2. Proof of the theorem

Let $N(T)$ denote, as usual, the number of non-trivial zeroes of $\zeta(\sigma + it)$ with $0 \leqslant t \leqslant T$. Working directly from (1) it follows that

$$\sum_{n=1}^{K}\phi_n = \sum_{n=1}^{K}\left(n - \frac{3}{2}\right) - \pi^{-1}\sum_{n=1}^{K}\theta(\gamma_n).$$

Since it can be verified[2] that $\theta(c) = 0$ for $c \approx 17.3\ldots$, write

[2] Indeed, $\theta(0) = 0$; $\theta(t)$ is decreasing for $0 < t < c$ whereafter $\theta(t)$ is monotonically increasing.

$$\theta(\gamma_n) = \int_c^{\gamma_n} \theta'(t)\,dt.$$

The range of integration is taken beyond $t = 0$ to avoid future difficulties with the evaluation of logarithmic terms in the integrand. Then

$$\sum_{n=1}^{K} \phi_n = \frac{K(K-2)}{2} - \pi^{-1} \sum_{n=1}^{K} \int_c^{\gamma_n} \theta'(t)\,dt,$$

and the order of summation and integration can be inverted leading to

$$\begin{aligned}
\sum_{n=1}^{K} \phi_n &= \frac{K(K-2)}{2} - \pi^{-1} \int_c^{\gamma_K} \theta'(t) \left(\sum_{n=1,\ \gamma_n \geqslant t}^{K} 1 \right) dt \\
&= \frac{K(K-2)}{2} - \pi^{-1} \int_c^{\gamma_K} \theta'(t) \bigl\{ N(\gamma_K) - N(t) \bigr\}\,dt \\
&= \frac{K(K-2)}{2} - \pi^{-1} K\theta(\gamma_K) + \pi^{-1} \int_c^{\gamma_K} \theta'(t) N(t)\,dt. \qquad (3)
\end{aligned}$$

Now using (see, e.g. [1, p. 173])

$$N(t) = S(t) + \pi^{-1}\theta(t) + 1, \qquad (4)$$

one can rewrite the integral in (3) as

$$\pi^{-1} \int_c^{\gamma_K} \theta'(t) N(t)\,dt = \pi^{-1} \int_c^{\gamma_K} \theta'(t) S(t)\,dt + \frac{\{\theta(\gamma_K)\}^2}{2\pi^2} + \theta(\gamma_K),$$

after integrating termwise and using $\theta(c) = 0$. Thus

$$\sum_{n=1}^{K} \phi_n = \frac{K(K-2)}{2} - \frac{\theta(\gamma_K)(K-1)}{\pi} + \frac{\{\theta(\gamma_K)\}^2}{2\pi^2} + \pi^{-1} \int_c^{\gamma_K} \theta'(t) S(t)\,dt.$$

Applying (4) once more with $t = \gamma_K$ one finds that

$$\sum_{n=1}^{K} \phi_n = -\frac{1}{2} + \frac{1}{2} \bigl\{ S(\gamma_K) \bigr\}^2 + \pi^{-1} \int_c^{\gamma_K} \theta'(t) S(t)\,dt. \qquad (5)$$

For $t > 0$ one has the estimate

$$\theta'(t) = \frac{1}{2} \log \frac{t}{2\pi} + O\bigl(t^{-2}\bigr), \qquad (6)$$

and indeed sharp estimates of the implicit constant can be found in [5]. By the second mean-value theorem for integrals, or by integrating by parts, it follows from (6) that

$$\begin{aligned}\int_c^{\gamma_K} \theta'(t) S(t)\,dt &= O\left(\log \gamma_K \int_c^{\gamma_K} S(t)\,dt\right) + O\left(\max_{c \leqslant \tau \leqslant \gamma_K} \left|\int_c^{\tau} S(t)\,dt\right|\right) \\ &= O\left(\log^2 \gamma_K\right),\end{aligned} \tag{7}$$

by using the well-known result of Littlewood on the function $S(t)$, viz. $\int_0^T S(t)\,dt = O(\log T)$. The confluence of Eqs. (7) and (5) and the estimate $S(T) = O(\log T)$ is

$$\sum_{n=1}^{K} \phi_n = -\frac{1}{2} + O\left(\log^2 \gamma_K\right).$$

Since the Riemann–von-Mangoldt formula gives $N(\gamma_K) = K \sim \frac{\gamma_K}{2\pi} \log \frac{\gamma_K}{2\pi}$, it follows that $\log K \sim \log \gamma_K$, and hence that

$$\sum_{n=1}^{K} \phi_n \ll \log^2 K, \tag{8}$$

whence the result in the theorem.

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References

[1] H.M. Edwards, Riemann's Zeta Function, Pure Appl. Math., Academic Press, New York, 1974.
[2] A. Fujii, On a conjecture of Shanks, Proc. Japan Acad. Ser. A 70 (4) (1994) 109–114.
[3] C. Haselgrove, J.C.P. Miller, Tables of the Riemann Zeta Function, Royal Society Mathematical Tables, vol. 6, Cambridge University Press, New York, 1960.
[4] W.D. Joyner, Distribution Theorems of L-Functions, Pitman Res. Notes Math., John Wiley and Sons, Inc., New York, 1986.
[5] R.S. Lehman, On the distribution of zeros of the Riemann zeta-function, Proc. Lond. Math. Soc. 3 (20) (1970) 303–320.
[6] A. Selberg, The zeta-function and the Riemann hypothesis, in: C. R. Dixième Congrès Math. Scandinaves, Copenhagen, 1946, pp. 187–200.
[7] D.S. Shanks, Review of 'Tables of the Riemann zeta function by C.B. Haselgrove and J.C.P. Miller', Math. Comp. 15 (73) (1961) 84–86.
[8] E.C. Titchmarsh, The zeros of the Riemann zeta-function, Proc. Roy. Soc. Ser. A 151 (1935) 234–255.
[9] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2nd edition, Oxford Science Publications, Oxford University Press, Oxford, 1986.
[10] T. Trudgian, Gram's Law fails a positive proportion of the time, arXiv:0811.0883, 2008.